Algebraic Number Theory

Exercise Sheet 12

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Exercise 1. Questions (3) - (6) from Exercise 3, Sheet 11.

Exercise 2. (1) Show that the field $K = \mathbb{Q}(i, \sqrt{3})$ is a Galois extension of $\mathbb{Q}$ of degree 4.
(2) Using Theorem 15 give the structure of the unit group of $\mathcal{O}_K$.

Exercise 3. Let $L = \mathbb{Q}(i, \sqrt{5})$.
(1) Show that L is a Galois extension of $\mathbb{Q}$ of degree 4.
(2) Let A be a principal ideal domain, K its field of fractions, L a separable finite extension of K of degree n, B the integral closure of A in L. Suppose that for a family $\{x_1, ..., x_n\}$ in B, the discriminant $D_A^B(x_1, ..., x_n)$ is square-free in A. Then show that the family $\{x_1, ..., x_n\}$ is a basis of B over A.
(4) Use (2) for $A = \mathbb{Z}[i]$, $B = \mathcal{O}_L$ and for the family $\{1, (1 + \sqrt{5})/2\}$ to find a basis of $\mathcal{O}_L$ over $\mathbb{Z}[i]$. Deduce that the ring of integers $\mathcal{O}_L$ is $\mathbb{Z}[i, (1 + \sqrt{5})/2]$.
(5) Let $K = \mathbb{Q}(\sqrt{5}i)$. Show that $K \subset L$, and that no prime ideal of $\mathcal{O}_K$ ramifies in L.
Hint: Show that $\mathcal{D}_{\mathcal{O}_K}^{\mathcal{O}_L} = \mathcal{O}_K$.
(6) Let p be an odd prime number in $\mathbb{Z}$, such that -1 is not a square modulo p. For the decomposition of $p\mathcal{O}_L$ compute corresponding e and f. Show that p does not ramify in L.
Hint: Use the description of $\mathcal{O}_L$ from (4), to show that
$\mathcal{O}_L \simeq \mathbb{Z}[X, Y]/(X^2 + 1, Y^2 - Y - 1)$. Then consider $\mathcal{O}_L/(p)$.